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A reconsideration of Hicks' non-linear trade cycle model

Cars H. Hommes

Department of Economic Statistics, University of Amsterdam, Roetersstraat 11, 1018 WB Amsterdam, The Netherlands

Abstract

The dynamics of Hicks' classical non-linear trade cycle model is reconsidered. The model is an unstable multiplier–accelerator model with an 'income ceiling' and an 'investment floor'. In the simplest two-dimensional version of the model (the basic Hicks model), all consumption lags one period behind income, while all investment outlays are concentrated in the period immediately following the originating change in income. We show that, in addition to stable cycles, so-called 'quasi-periodic attractors' can occur in the basic Hicks model. However, in the simplest version of the model, the dynamics are always regular and chaotic income fluctuations do not occur. When consumption and/or investment is distributed over several time periods, higher dimensional versions of the Hicks model are obtained, and the dynamic behaviour can be much more complicated. We show that in the simplest version, the three-dimensional Hicks model, strange-chaotic attractors can occur. The three-dimensional Hicks model also exhibits coexistence of different attractors.

Keywords: Endogenous business cycles; Ceiling and floor model; Strange attractors

JEL classification: E32; E12

1. Introduction

The evidence adduced to date should make us suspicious of 'Frischian' models. There is enough contrary evidence now that we should be investigating non-linear models. It has taken almost 50 years for this particular idea of Hicks to catch on (William A. Brock, 1991). In the last two decades there has been a rapidly growing interest in non-linear dynamic models, both in mathematics and in applied sciences such as physics, biology, meteorology and chemistry. In particular, the discovery that simple non-linear models may exhibit complicated, chaotic dynamics has received

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much attention. In economics there is also a growing interest in non-linear dynamic models. Examples of non-linear economic models that exhibit chaotic dynamics have, for example, been provided by Benhabib and Day (1981, 1982), Day (1982, 1983), Grandmont (1985), and Boldrin and Montrucchio (1986). For surveys on chaos and economics, see, for example, Baumol and Benhabib (1989), Lorenz (1993), Boldrin and Woodford (1990), Brock *et al.* (1991), and Medio (1992).

Interest in non-linear dynamic models in economics is not new, however, and dates back to the time before economists had learned about chaos. Kaldor (1940), Hicks (1950), and Goodwin (1951) have already tried to model economic fluctuations by non-linear deterministic business cycle models. At that time, attention was focused on regular, periodic dynamic behaviour rather than on irregularity and chaos. In this paper, we study one of the first and one of the simplest non-linear business cycle models, introduced by Hicks (1950). In particular, we concentrate on the following question: *Does each time path in Hicks' non-linear trade cycle model converge to a periodic time path*? Although the Hicks model can be found in many textbooks on economic dynamics, it seems that this question has not yet been answered.

Hicks' non-linear trade cycle model is an unstable linear multiplier–accelerator model with a 'ceiling' (full employment upper bound) and a 'floor' (investment lower bound). It is a piecewise linear dynamic model that generates cycles. The simplest version is a two-dimensional model. As we will see, even for the simplest version (the basic Hicks model), a time path does not always converge to a periodic time path. In addition to stable periodic orbits, the basic Hicks model can have a so-called 'quasi-periodic attractor'. In that case, all time paths are aperiodic and do not converge to a periodic time path. However, we show that, in the basic Hicks model, the dynamics are always regular. Income fluctuations are either periodic or almost periodic (quasi-periodic). Irregular, chaotic income fluctuations do not occur in the simplest version of the Hicks model.

In the basic Hicks model, all consumption lags one period behind income, while all induced investment is concentrated in the period immediately following the originating change in income. Hicks (1950, p.73) has already posed the following questions: (1) *What difference does it make when some consumption is lagged more than one period behind income*? (2) *What difference does it make if induced investment is postponed or distributed over a number of time periods*?

When consumption is distributed over three time periods following income, and induced investment is distributed over two time periods following the change in income, the Hicks model is a three-dimensional (3D) piecewise linear model. The dynamics of the 3D Hicks model can be very complicated. We show that the 3D Hicks model can have strange, chaotic attractors. The pictures of these attractors suggest a 'Cantor-like' structure and the largest Lyapunov exponent is positive. The 3D Hicks model also exhibits another phenomenon related to complex dynamics; namely, coexistence of different attractors, for example, coexistence of a strange, chaotic attractor and a stable periodic orbit. In that case the initial state determines whether income fluctuations will be irregular.

The paper is organized as follows. Section 2 reconsiders the dynamics of the basic Hicks model. In Section 3, we investigate the 3D Hicks model with distributed

consumption and distributed investment. Section 4 contains some concluding remarks. Some mathematical details and proofs are in the appendices.

2. The basic 2-D Hicks model

We reconsider the simplest version of Hick's non-linear trade cycle model in three subsections. Subsection 2.1 recalls the Hicks model. A geometric description of the dynamics is given in Subsection 2.2, whereas Subsection 2.3 describes the main results concerning the dynamics.

2.1. The model

Hicks' non-linear trade cycle model is perhaps the simplest non-linear model of the business cycle. The model describes the dynamic behaviour of national income, consumption and investment. Hicks' original trade cycle model is a growth model, with a unique unstable equilibrium growth path, a maximum growth path (the full employment 'ceiling') and a minimum growth path (the zero investment 'floor'), all with the same constant growth rate. The time paths in the original Hicks model can be described as oscillations between the 'floor' and the 'ceiling', around a constant equilibrium trend growth rate (Hicks, 1950, or, for example, Gandolfo, 1985).

The model that we will consider is a slightly modified version obtained from Hicks' original trade cycle model by removing the constant trend growth rate, see Rau (1974) and Blatt (1983). In the modified version, there exists a unique unstable equilibrium, and all time paths are bounded. The variables of the model are income (output) Y_t, consumption C_t and investment I_t, where t denotes the time period (e.g. a quarter of a year). In fact, Hicks' non-linear trade cycle model is simply the well-known linear multiplier–accelerator model with a 'ceiling' and a 'floor'. The linear multiplier–accelerator model is given by the following four equations:

$$\text{Consumption:} \quad C_t = mY_{t-1} \tag{1}$$

$$\text{Total investment:} \quad I_t = I_t^{\text{ind}} + I^{\text{aut}}, \tag{2}$$

$$\text{Induced investment:} \quad I_t^{\text{ind}} = a(Y_{t-1} - Y_{t-2}), \tag{3}$$

$$\text{Income:} \quad Y_t = C_t + I_t. \tag{4}$$

According to (1), current consumption is proportional to the previous income, with m the *marginal propensity to consume*, $0 < m < 1$. Investment consists of constant autonomous investment I^{aut} plus induced investment I_t^{ind}, which is proportional to the growth in national income, and a is called the *accelerator*. Finally, according to (4), income equals consumption plus investment. The linear multiplier–accelerator model of (1–4) is closely related (but not identical) to the original linear multiplier–accelerator model (Samuelson, 1939; cf. Gandolfo, 1985). Substituting (1–3) into (4), a linear second-order difference equation that describes the dynamics of income Y_t is obtained:

$$Y_t - (m + a)Y_{t-1} + aY_{t-2} = I^{\text{aut}}. \tag{5}$$

The unique equilibrium $Y^e = I^{aut}/(1 - m)$ is stable if and only if the accelerator $a < 1$. However, on the basis of ecomomic data, Hicks observed that $a > 1$, so that the equilibrium is unstable. For $a > 1$, the multiplier–accelerator model of (1)–(4) generates exploding time paths. To prevent unbounded oscillations, Hicks imposed a 'ceiling' (an income upper bound) and a 'floor' (an investment lower bound) on the unstable linear model. More precisely, Hicks replaced the linear equations (3) and (4) by two piecewise linear equations:

$$\text{Investment floor:} \qquad I_t^{ind} = \max\{a(Y_{t-1} - Y_{t-2}), -I^f\}, \tag{3a}$$

$$\text{Income ceiling:} \qquad Y_t = \min\{C_t + I_t, Y^c\}. \tag{4a}$$

The economic considerations leading to these non-linear equations were the following. When income is decreasing, then, according to (3), induced investment becomes negative. However, falls in output do not induce disinvestment in the same way as rises in output induce investment. It is natural to assume that the negative net investment is bounded by some negative lower bound $-I^f$, equal to the maximum amount of depreciation of the corresponding capital stock. This motivates Eq. (3a). The second non-linear equation (4a) was motivated by the observation that, in any economy, income is limited by the scarcity of employable resources (e.g. labour, land, raw materials, capital and productivity); therefore, a ceiling level of employment Y^c is postulated in Eq. (4a). Whenever the demand for output, i.e. consumption plus investment, exceeds Y^c, the economy cannot meet the total demand, and income Y^t will assume its ceiling level Y^c.

We will call the model given by (1), (2), (3a) and (4a) the *basic Hicks model*. Given the initial values Y_{-1} and Y_0, all future values of consumption C_t, investment I_t and income Y_t, $t \geq 1$, are completely determined. Recall that the linear multiplier–accelerator model of (1)–(4) has a unique equilibrium $Y^e = I^{aut}/(1 - m)$. We assume $Y^e < Y^c$ (income ceiling) and that the accelerator $a > 1$. The basic Hicks model then has a unique, unstable equilibrium Y^e. However, the 'income ceiling' and the 'investment floor' prevent unbounded time paths.

Let us briefly recall the economic intuition behind the dynamics of the model, as described by Hicks (1950). We start from the initial states $Y_{-1} = Y_0 = Y^c$, i.e. income starts at its ceiling level. With income at its ceiling level for two consecutive time periods, the induced investment becomes zero ($I_1^{ind} = 0$), causing income to fall, since $Y_1 = mY_0 + I^{aut} = mY^c + I^{aut} < Y^c = Y_0$, where the $<$ follows from $Y^e < Y^c$ and $m < 1$. A 'downswing' of the economy begins. When income decreases, induced investment becomes negative, and may reach its floor level. With induced investment at its floor level, income will continue to decrease. However the decrease becomes smaller and smaller, and sooner or later induced investment will exceed its floor level, causing income to rise again: an 'upswing' of the economy begins. When income increases induced investment becomes positive, reinforcing the rise in income. The 'upswing' of the economy may continue, until income reaches its ceiling level again. When income is at its ceiling level for two consecutive time periods, induced investment becomes zero agin, causing a new 'downswing' of the economy, etc. This interaction between the investment floor in (3a) and the income ceiling in (4a) leads

to oscillatory behaviour of both income and investment. Observe that the two non-linearities are quite different in nature. The ceiling Y^c is a direct barrier upon the expansion of income. However, the investment floor provides an indirect check of a decreasing income, which sooner or later will cause income to rise again.

We emphasize that Hicks' description of the economic mechanism of the non-linear trade cycle model is only an intuitive description. Such a description is certainly useful, since it provides insight into how the model works. Hicks' description, based on economic intuition, explains why this non-linear trade cycle model generates oscillatory behaviour of income and investment. However, Hick's intuitive description leaves open the following fundamental question: Does a time path in Hicks' non-linear trade cycle model always converge to a *periodic* time path?

2.2. Geometric description of the dynamics

To gain insight into the dynamics, we first investigate the model from a geometric point of view. It seems that such a geometric approach is still lacking, although our approach shows some similarities with the treatment of the Hick model in Baumol (1958). Substituting (1), (2) and (3a) into (4a), we obtain a piecewise linear second-order difference equation, describing the dynamics of income Y_t:

$$Y_t = \min\{mY_{t-1} + \max\{a(Y_{t-1} - Y_{t-2}), -I^f\} + I^{aut}, Y^c\}. \tag{6}$$

Using $x_t = Y_t$ and $y_t = Y_{t-1}$, we obtain the following two-dimensional system:

$$\begin{aligned} x_{t+1} &= \min\{mx_t + \max\{a(x_t - y_t), -I^f\} + I^{aut}, Y^c\}, \\ y_{t+1} &= x_t. \end{aligned} \tag{7}$$

Consequently, the basic Hicks model is determined by a difference equation $(x_{t+1}, y_{t+1}) = H(x_t, y_t)$, where H is the two-dimensional piecewise linear map:

$$H(x, y) = (\min\{mx + \max\{a(x - y), -I^f\} + I^{aut}, Y^c\}, x). \tag{8}$$

We define the income floor level $Y^f = (I^{aut} - I^f)/(1 - m)$ and the domain of the map H as $D = \{(x, y) | Y^f \leq x, y \leq Y^c\}$. A simple computation shows that H maps the domain D into itself. Consequently, all time paths in the model are bounded. If the initial states Y_{-1} and Y_0 are between Y^f and Y^c, then income Y_t will always be between the floor level Y^f and the ceiling level Y^c. We divide the domain D into the three subdomains D^f, D^l and D^c (see Fig. 1):

$$\text{Floor region:} \quad D^f = \{(x, y) \in D | a(x - y) < -I^f\}, \tag{9a}$$

$$\text{Ceiling region:} \quad D^c = \{(x, y) \in D | mx + a(x, y) + I^{aut} > Y^c\}, \tag{9b}$$

$$\text{Linear region:} \quad D^l = D \setminus (D^f \cup D^c). \tag{9c}$$

Note that, when a point (x_0, y_0) lies in the floor region, then, in the *next* time period, induced investment reaches its floor level, i.e. $I_1^{ind} = -I^f$. Similarly, when a point (x_0, y_0) lies in the ceiling region, then, in the *next* time period, income reaches its

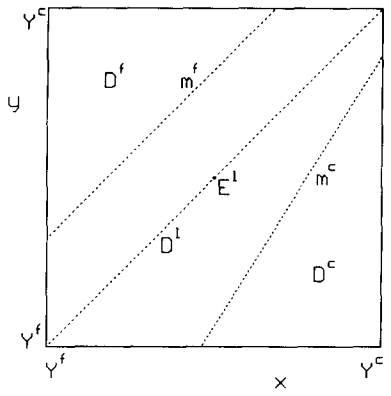


Fig. 1. The floor region D^f, the ceiling region D^c, the linear region D^l, the floor boundary line m^f and the ceiling boundary line m^c. The piecewise linear map H is linear in each of the regions D^f, D^c and D^l.

ceiling level, i.e. $Y_1 = Y^c$. The boundary lines between the linear region and the floor region and the ceiling region, respectively, are:

$$\text{Floor boundary line:} \quad m^f = \{(x, y) \in D | y = x + I^f/a\}, \tag{10a}$$

$$\text{Ceiling boundary line:} \quad m^c = \{(x, y) \in D | y = (1 + m/a)x - (Y^c - I^{aut})/a\}. \tag{10b}$$

The continuous, piecewise linear map H is linear in each of the three regions D^f, D^l and D^c. The corresponding linear maps H^f, H^l and H^c are:

$$\text{Floor map:} \quad H^f(x, y) = (mx + I^{aut} - I^f, x), \tag{11a}$$

$$\text{Multiplier–accelerator map:} \quad H^l(x, y) = (mx + a(x - y) + I^{aut}, x), \tag{11b}$$

$$\text{Ceiling map:} \quad H^c(x, y) = (Y^c, x). \tag{11c}$$

The map H^c has a stable equilibrium $E^c = (Y^c, Y^c)$, with a double eigenvalue 0. Observe that the equilibrium E^c lies in the region D^l, and not in D^c, so that E^c is not an equilibrium of the piecewise linear map H. The map H (or H^c) maps the region D^c into the line $x = Y^c$, and the line $x = Y^c$ into the line $y = Y^c$. The linear map H^f has an equilibrium $E^f = (Y^f, Y^f)$, where Y^f is the income floor level. The eigenvalues of the linear map H^f are 0 and m. Since $0 < m < 1$, we conclude that H^f has a stable equilibrium E^f. Observe that E^f lies in D^l, and not in D^f, so that E^f is not an equilibrium of the piecewise linear map H. Since H^f has an eigenvalue 0, H (or H^f) maps the region D^f into the line segment $H(m^f) = H^f(m^f)$ with slope $1/m$ and containing the point E^f. It is important to note that, since the map H has one and two eigenvalues 0 in the regions D^f and D^c, respectively, *in both the floor region and the ceiling region, the Hicks model is, in fact, a one dimensional system.* Finally, since the accelerator $a > 1$, the linear map H^l has an unstable equilibrium $E^l = (Y^e, Y^e)$, with either complex eigenvalues, or real eigenvalues larger than 1.

Summarizing, the dynamics of the basic Hicks model are described by a difference equation $(x_{n+1}, y_{n+1}) = H(x_n, y_n)$, where H is a continuous, piecewise linear map. As

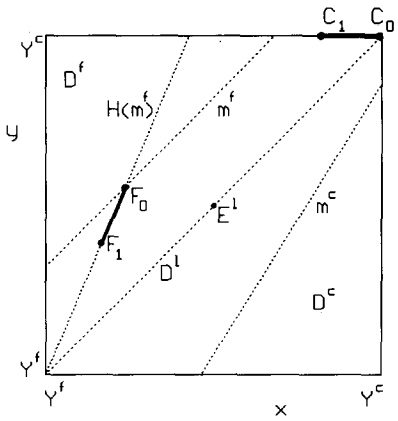


Fig. 2. The ceiling segment C_0C_1 and the floor segment F_0F_1. These two line segments completely determine the long-run dynamics of the basic Hicks model.

long as the state (x, y) lies in the floor region D^f, then H is a linear contraction towards the point E^f. With (x, y) in the ceiling region D^c, H is a linear contraction, with double eigenvalue 0, towards the point E^c. When the point (x, y) lies in the linear region D^l, then H is a linear expanding map, with both eigenvalues outside the unit circle. Therefore, a time path with an initial state not equal to the equilibrium E^l cannot be contained in the region D^l. Each time path (except for the equilibrium) has points in the floor region D^f or in the ceiling region D^c.

Define $C_0 = (Y^c, Y^c)$ and $C_1 = H(C_0)$. We call the line segment C_0C_1 the *ceiling segment*; see Fig. 2. Recall that H maps the ceiling region D^c into the vertical line through the point C_0, and this vertical line into the horizontal line through C_0. For each initial state (x, y) in the ceiling region D^c, the point $H^2(x, y)$ lies in the ceiling segment. The map H maps the floor region D^f into the line $H(m^f)$. Let F_0 be the intersection point of the lines m^f and $H(m^f)$, and let $F_1 = H(F_0)$. We call the line segment F_0F_1 the *floor segment*; see Fig. 2. It is straightforward to show that, for each initial state (x, y) in the floor region D^f, there exists a smallest positive integer n such that $H^n(x, y)$ either lies in the ceiling segment or in the floor segment. Hence, the long-term dynamics of the basic Hicks model are completely determined by the floor segment and the ceiling segment.

2.3. *Periodic and quasi-periodic dynamics*

This section describes the main results concerning the dynamics of the basic Hicks model. Here, we will not go into too much technical detail, but rather provide an intuitive description. Some mathematical details are given in Appendix A1; additional details and proofs can be found in Hommes (1991).

The first result is that the basic Hicks model has a *one-dimensional attracting set*, which is a piecewise linear closed curve encircling the unstable equilibrium (cf. Fig. 3(a)). All income time paths, except for the unstable equilibrium, are attracted to this attracting set. In fact, as soon as a time path 'hits' the ceiling or the floor, i.e. as soon

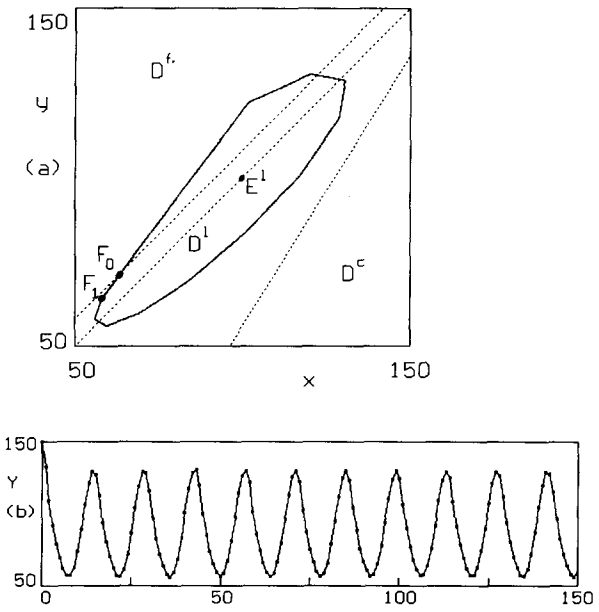


Fig. 3. (a) Quasi-periodic attractor in the basic Hicks model with $Y^c = 150$, $I^f = 10$, $I^{aut} = 20$, $m = 0.75$ and $a = 1.25$. (b) The corresponding income time path, although not exactly periodic, is almost periodic.

as income assumes its ceiling level or induced investment assumes its floor level, the time path enters the attracting set. In the phase space, the attracting set is obtained by iterating the floor segment and/or ceiling segment.

The long-run dynamics are completely determined by the dynamics on the attracting set. Mathematically, the dynamics on the attracting set are equivalent to a continuous, piecewise linear, non-decreasing, degree 1 circle map. The most important corollary of this is that the time paths in the basic Hicks model can be described as a (counter-clockwise) cyclic movement with a unique average rotation around the unstable equilibrium. We call this unique average rotation the *rotation number* of the basic Hicks model. When the rotation number is rational, then all time paths converge to a (stable) periodic orbit. However, the rotation number can be irrational. In that case, the time paths (except for the equilibrium) are aperiodic; i.e. the time paths are not periodic and they do not converge to a periodic time path. In particular, the basic Hicks model can have a *quasi-periodic attractor*. An example of a quasi-periodic attractor (or a periodic attractor with a very long period) is shown in Fig. 3(a). The quasi-periodic attractor is a piecewise linear closed curve containing the floor segment F_0F_1.[1] All orbits, except for the equilibrium, are dense in the

[1] The quasi-periodic attractor in Fig. 3(a) has no points in the ceiling region D^c, so that income never reaches its ceiling level. If income does reach its ceiling level, then quasi-periodic behaviour, although not impossible, is less likely to occur. A stable periodic orbit is then more likely, since the map H has a double eigenvalue 0 in the ceiling region. In particular, a stable cycle containing the point $(x, y) = (Y^c, Y^c)$, with income for two consecutive periods at the ceiling level, occurs for many parameter constellations.

attractor. The corresponding income time path (Fig. 3(b)), although not exactly periodic, is almost periodic.

Summarizing, although aperiodic time paths may occur, the dynamics in the basic Hicks model are always regular: all time paths rotate with a unique average rotation around the unstable equilibrium. The movement around the unstable equilibrium is either periodic or quasi-periodic, so that the income fluctuations are either periodic or almost periodic. Chaotic dynamics and sensitivity to initial states do not occur in the simplest versions of Hicks' non-linear trade cycle model.

3. The 3D Hicks model

In Section 2, we have analyzed the dynamics of the basic Hicks model, which is, in fact, just the simplest form of the (general) Hicks model. In the basic Hicks model, all consumption lags one period behind income, while all induced investment outlays are concentrated in the time period immediately following the originating change in income. Hicks (1950) has already introduced a more general form of the model, in which both consumption and induced investment are distributed over several time periods. The dynamics of the basic Hicks model are always regular (periodic or quasi-periodic) and chaos does not occur. In this section, we investigate whether or not this regularity in the dynamics also holds for the general Hicks model. In particular, we consider the following fundamental problem: *can Hicks' non-linear trade cycle model with distributed consumption – and investment – lags, generate chaotic time paths?*

We investigate this problem for the 3D version of the Hicks model, in which the consumption is distributed over the three time periods following income, while induced investment is distributed over two time periods following the change in income. The 3D Hicks model is the second simplest version of the Hicks model.

This section is subdivided into three subsections. The 3D Hicks model is presented in Subsection 3.1. In Subsection 3.2., we recall some definitions related to the notion of strange attractor. Finally Subsection 3.3 contains numerical results concerning the dynamics of the 3D Hicks model

3.1. The 3D model

In the general Hicks model, consumption and induced investment are spread over several time periods. The general Hicks model is an nth order, or equivalently, an n-dimensional, piecewise linear difference equation. We will concentrate on the second simplest version of the model, the 3D Hicks model. In the 3D version, distributed consumption and distributed induced investment are given by the linear equations:

$$\text{Distributed consumption:} \quad C_t = m_1 Y_{t-1} + m_2 Y_{t-2} + m_3 Y_{t-3}, \tag{12a}$$

$$\text{Distributed induced investment:} \quad I_t^{\text{ind}} = a_1(Y_{t-1} - Y_{t-2}) + a_2(Y_{t-2} - Y_{t-3}). \tag{12b}$$

The parameters m_i, $1 \le i \le 3$, are the *partial consumption coefficients*, $0 < m_i < 1$, determining which fraction of income is consumed i time periods later. The sum $m = m_1 + m_2 + m_3$ is the *total marginal propensity to consume*, $0 < m < 1$. The parameters a_i, $i = 1, 2$, are the *partial investment coefficients*, $a_i \ge 0$, determining how much the current induced investment is affected by the change in income $(Y_{t-i} - Y_{t-i-1})$, which occurred i time periods ago. The sum $a = a_1 + a_2$ is the *total investment coefficient* or the *total accelerator*. Observe that in the most general 3D version we have three partial consumption coefficients, m_1, m_2 and m_3, but only two partial investment coefficients, a_1 and a_2. With distributed consumption as in (12a) and distributed induced investment as in (12b), the 3D *Hicks model*[2] is given by

$$C_t = m_1 Y_{t-1} + m_2 Y_{t-2} + m_3 Y_{t-3}, \tag{13a}$$

$$I_t = I_t^{\text{ind}} + I^{\text{aut}}, \tag{13b}$$

$$I_t^{\text{ind}} = \max\{a_1(Y_{t-1} - Y_{t-2}) + a_2(Y_{t-2} - Y_{t-3}), -I^{\text{f}}\}, \tag{13c}$$

$$Y_t = \min\{C_t + I_t, Y^{\text{c}}\}. \tag{13d}$$

We will investigate the following problem: *What can be sad about the dynamics of the 3D Hicks model*? First, we generalize some notations of Subsection 2.2. Substituting (13a)–(13b) into (13d) with $x_t = Y_t$, $y_t = Y_{t-1}$ and $z_t = Y_{t-2}$, the 3D Hicks model is reduced to a piecewise linear difference equation:

$$\begin{aligned} x_{t+1} &= \min\{m_1 x_t + m_2 y_t + m_3 z_t + \max\{a_1(x_t - y_t) + a_2(y_t - z_t), -I^{\text{f}}\} + I^{\text{aut}}, Y^{\text{c}}\}, \\ y_{t+1} &= x_t, \\ z_{t+1} &= y_t. \end{aligned} \tag{14}$$

Hence, the 3D Hicks model is given by a difference equation $(x_{t+1}, y_{t+1}, z_{t+1}) = H(x_t, y_t, z_t)$, where H is the 3D continuous piecewise linear map:

$$\begin{aligned} H(x, y, z) = (&\min\{m_1 x + m_2 y + m_3 z + \max\{a_1(x - y) + a_2(y - z), -I^{\text{f}}\} \\ &+ I^{\text{aut}}, Y^{\text{c}}\}, x, y). \end{aligned} \tag{15}$$

Write $Y^{\text{e}} = I^{\text{aut}}/(1 - m)$, $m = m_1 + m_2 + m_3$ and assume $Y^{\text{e}} < Y^{\text{c}}$. The map H has a unique equilibrium $E^{\text{e}} = (Y^{\text{e}}, Y^{\text{e}}, Y^{\text{e}})$. Define the income floor level $Y^{\text{f}} = (I^{\text{aut}} - I^{\text{f}})/(1 - m)$. Note that $Y^{\text{f}} < Y^{\text{e}} < Y^{\text{c}}$. The domain D of H is defined as $D = \{(x, y, z) | Y^{\text{f}} \le x, y, z \le Y^{\text{c}}\}$. A simple computation shows that H maps the domain D into itself, so all time paths are bounded, The domain D is subdivided into the *floor region*, D^{f}, *the ceiling region*, D^{c}, and the *linear region*, D^{l}, given by

$$D^{\text{f}} = \{(x, y, z) \in D | a_1(x - y) + a_2(y - z) < -I^{\text{f}}\}, \tag{16a}$$

$$\begin{aligned} D^{\text{c}} = \{&(x, y, z) \in D | m_1 x + m_2 y + m_3 z + a_1(x - y) + a_2(y - z) \\ &+ I^{\text{aut}} > Y^{\text{c}}\}, \end{aligned} \tag{16b}$$

[2] For $a_2 = 0$, $m_3 = 0$ and $m_2 \neq 0$, a 2-D version is obtained that is different from basic two-dimensional HIcks model of Section 2. In Hommes (1991), it is shown that this two-dimensional version, unlike the two-dimensional basic Hicks model, can generate chaotic dynamics, especially when $m_2 > m_1$.

$$D^{l} = D \backslash (D^{f} \cup D^{c}). \tag{16c}$$

The piecewise linear map H is linear in each of the three regions D^{f}, D^{l} and D^{c}. We call these linear maps respectively, the *floor map*, H^{f}, the *multiplier–accelerator map*, H^{l}, and the ceiling map, H^{c}. The linear maps are given by

$$H^{f}(x, y, z) = (m_1 x + m_2 y + m_3 z + I^{aut} - I^{f}, x, y), \tag{17a}$$

$$H^{l}(x, y, z) = (m_1 x + m_2 y + m_3 z + a_1(x - y) + a_2(y - z) + I^{aut}, x, y), \tag{17b}$$

$$H^{c}(x, y, z) = (Y^{c}, x, y). \tag{17c}$$

The ceiling map H^{c} has a triple eigenvalue 0. The eigenvalues of the floor map H^{f} are the solutions of the characteristic equation:

$$\lambda^3 - m_1 \lambda^2 - m_2 \lambda - m_3 = 0. \tag{18}$$

Using $0 < m_1 + m_2 + m_3 < 1$ it can be shown that all eigenvalues of H^{f} lie inside the unit circle, so that H^{f} is a contraction. The equilibrium E^{l} lies in the linear region D^{l} and the corresponding characteristic equation is

$$\lambda^3 - (m_1 + a_1)\lambda^2 - \lambda(m_2 - a_1 + a_2) + a_2 - m_3 = 0. \tag{19}$$

Observe that the condition $a_2 - m_3 > 1$ is a sufficient condition for the instability of the equilibrium.

3.1.1. A period 2 orbit

An important difference between the dynamics of the basic Hicks model and the 3D Hicks model is the occurrence of a period 2 orbit in the 3D model. To state a result concerning the existence of a period 2 orbit, we introduce the following conditions on the parameters:

$$(1 + m_1 - m_2 + m_3)/2 < a_2 - a_1 < 1 + m_1 - m_2 + m_3, \tag{20a}$$

$$I^{aut} + I^{f}\left[\frac{(a_2 - a_1 - m_1 - m_3)}{(1 + m_1 - m_2 + m_3 + a_1 - a_2)}\right] < (1 - m_1 - m_2 - m_3) Y^{c}. \tag{20b}$$

Proposition 1. *The map H in (15) has a period 2 orbit with one point in the floor region* D^{f} *and the other point in the linear region* D^{l}, *if and only if the parameters satisfy conditions (20a) and (20b).*

A proof of Proposition 1 is given in Appendix B. With different conditions on the parameter values, a similar result holds concerning the existence of a period 2 orbit with points in D^{l} and D^{c}, and a period 2 orbit with points in D^{f} and D^{c}. We will see that the period 2 orbit with points in D^{l} and D^{f} is important in order to understand the dynamics of the 3D Hicks model. Observe that, according to condition (20a), the second partial investment coefficient a_2 has to be larger than the first partial investment coefficient a_1. In particular, it follows immediately that the basic Hicks model does not have a period 2 orbit with points in D^{l} and D^{f}.

Given the regions in which the two points of a period 2 orbit lie, the characteristic equation determining the eigenvalues corresponding to that particular period 2 orbit

is easily obtained. For example, the characteristic equation corresponding to the period 2 orbit with points in D^l and D^f is simply the characteristic equation of the product of the Jacobian matrices of the linear maps H^l and H^f. The three characteristic equations are:

$$\text{Period 2 with points in } D^f \text{ and } D^c: \quad \lambda^2(m_2 - \lambda) = 0 \tag{21a}$$

$$\text{Period 2 with points in } D^l \text{ and } D^c: \quad \lambda^2(m_2 - a_1 + a_2 - \lambda) = 0, \tag{21b}$$

$$\text{Period 2 with points in } D^f \text{ and } D^l: \quad \lambda^3 + c_1\lambda^2 + c_2\lambda + c_3 = 0, \tag{21c}$$

where

$$c_1 = -[m_1(m_1 + a_1) + 2m_2 - a_1 + a_2],$$

$$c_2 = -[m_1(m_3 - a_2) - m_2(m_2 - a_1 + a_2) + m_3(m_1 + a_1)]$$

and

$$c_3 = -m_3(m_3 - a_2).$$

Since $0 \leq m_2 < 1$, it follows from (21a) that the first period orbit (if it exists) is always stable. However, the second and third period 2 orbits may be unstable. We will see that, if the period 2 orbit with points in D^f and D^l is unstable, then the dynamics can be very complicated.

3.2. *Strange, chaotic attractors, Lyapunov exponents and dimension*

There are several definitions of the notion of a strange and chaotic attractor. We will use the definition based on Lyapunov exponents. The *largest Lyapunov exponent* of the map $f:\mathbb{R}^m \to \mathbb{R}^m$ (at the point x) is defined as[3]

$$\lambda_1 = \lim_{n\to\infty} \frac{1}{n} \log\|(D_x f^n)(\delta x)\|, \tag{22}$$

where $D_x f^n$ is the Jacobian matrix of the map f^n at the point x, δx is an initial perturbation vector and $\| \|$ denotes the length of a vector in $\mathbb{R}^m$. The largest Lyapunov exponent measures the largest average rate of divergence of nearby initial states. In general, for points x converging to a certain attractor, n Lyapunov exponents $\lambda_1 > \lambda_2 > \lambda_3 > \cdots > \lambda_n$ can be defined; for mathematical details, see, for example, Eckmann and Ruelle (1985) and Wolf (1986).

An attractor is called *strange* or *chaotic* when the largest Lyapunov exponent is positive. For a stable periodic orbit all Lyapunov exponents are negative, while for a quasi-periodic attractor homeomorphic to a circle the largest Lyapunov exponent is 0 and all other Lyapunov exponents are negative.

[3] The piecewise linear Hicks model is not differentiable on the boundaries between the floor, the ceiling and the linear regions. Since this is a zero Lebesque measure set, however, this does not cause problems for the definition of Lyapunov exponents.

An important geometric characterization of attractors is the dimension. Again, there exist several different notions of dimension. Here we use the Lyapunov dimension and the correlation dimension. The *Lyapunov dimension* D_L is

$$D_L = j + \frac{\sum_{i=1}^{j} \lambda_i}{|\lambda_{j+1}|}, \tag{23}$$

where j is the largest integer for which $\sum_{i=1}^{j} \lambda_i > 0$.Next define

$$C(\varepsilon) = \lim_{N \to \infty} \frac{\#\{(x_i, x_j) | \|x_i - x_j\| < \varepsilon, 1 \leq i, j \leq N\}}{N^2}, \tag{24}$$

where $\# A$ is the number of points in the set A. $C(\varepsilon)$ is the probability that two points x_i and x_j are within a distance ε. The *correlation dimension* D_C is

$$D_C = \lim_{\varepsilon \downarrow 0} \frac{\log C(\varepsilon)}{\log \varepsilon} \tag{25}$$

The correlation dimension measures the spatial correlation of points on an attractor. It has been found that generally $D_C \leq D_L$. For more details on fractal dimension and its numerical determination, see, for example, Grassberger and Procaccia (1983), Takens (1984), Farmer *et al.* (1983), Falconer (1990) and the discussion of dimension in Brock (1986).

In Subsection 3.3, for the 3D Hicks model we use the DYNAMICS-program (Nusse and York, 1992) to compute the Lyapunov exponents and Lyapunov dimension of the attractors, based on 10000 observations or more. The correlation dimensions for the 3D Hicks model are computed by using the chaos data analyzer (Sprott and Rowlands, 1992), based on 10000 observations.

3.3. Numerical simulations

In this Subsection we present the numerical result concerning the dynamics of the 3D Hicks model.[4] The partial consumption and the partial investment coefficients determine how consumption and induced investment are distributed over the different time periods. We concentrate on the case that seems to be the most relevant from an economic point of view. We assume $m_1 > m_2 > m_3 > 0$, so that the highest fraction of income is consumed with a delay of one time period, while the lowest fraction of income is consumed with a delay of three periods. Furthermore, we assume that $a_2 > a_1$, so that most of the induced investment takes place in the second time period following the originating change in income. As, for example, pointed out by Allen (1965), such an investment pattern may be the rule rather than the exception, since the time needed for investment decisions and investment outlays is usually longer than the shortest consumption lag. Therefore, in the 3D Hicks model, it seems to be reasonable to assume that most induced investment occurs with a delay of two time periods, i.e. $a_2 > a_1$.

[4] Some different numerical simulations can be found in Hommes (1993); extensive numerical simulations are given in Hommes (1991).

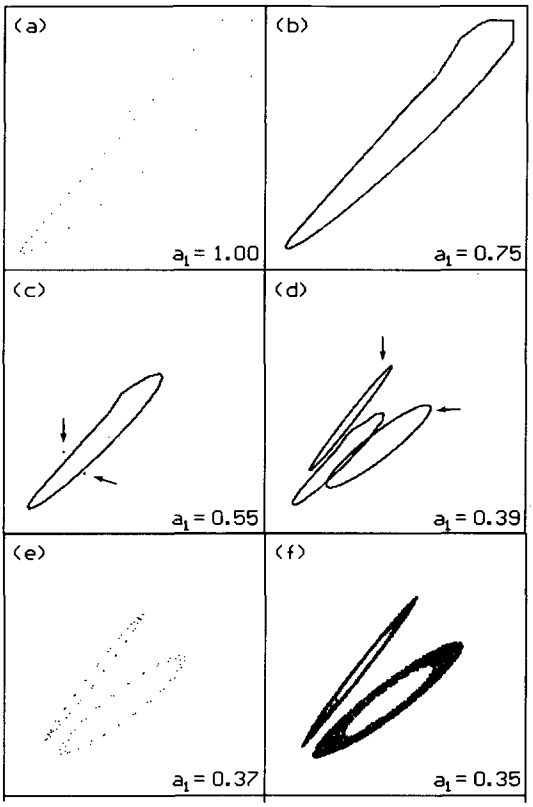


Fig. 4.(a)–(f) Horizontal x–axis: $80 \leq x \leq 260$. Vertical y–axis: $80 \leq y \leq 260$. Parameters: $Y^c = 250$, $I^f = 10$, $I^{aut} = 20$, $m_1 = 0.5$, $m_2 = 0.2$, $m_3 = 0.14$ and $a_2 = 1.31$. Attractors for decreasing values of a_1. (*a*) stable period 27 orbit: (*b*) quasi-periodic attractor; (*c*) coexistence of a stable period 2 orbit and a quasi-periodic attractor; (*d*) coexistence of two quasi-periodic attractors; (*e*) stable periodic orbit; (*f*) strange-chaotic attractor.

For clarity of presentation, we fix all parameters but one and present some typical phenomena, which occur for other parameter constellations as well. We fix all parameters, except the first partial investment coefficient a_1, as follows: $m_1 = 0.5$, $m_2 = 0.2$, $m_3 = 0.14$, $a_2 = 1.31$, $Y^c = 250$, $I^f = 10$ and $I^{aut} = 20$. We investigate how the dynamics of the model depend on the parameter a_1, $0 \leq a_1 < 1$. Observe that $a_1 < a_2$, $m_1 > m_2 > m_3$ and the overall marginal propensity to consume $m = 0.84$.

3.3.1. Phase–space plots of the attractors

Diagrams of the (x, y) projections of attractors for decreasing values of the partial investment coefficient a_1 are shown in Fig. 4(a–r). All the diagrams have been obtained by plotting a finite part (say, 5000 points) of a projected orbit, after a transient time of, say, 50 periods. For $a_1 = 1$ (Fig. 4(a)), a stable period 27 orbit occurs. For $a_1 = 0.75$ (Fig, 4(b)), the attractor seems to be a piecewise linear closed curve. The largest Lyapunov exponent is close to zero ($\lambda_1 \approx 0.000\,01$), indicating that we have a quasi-periodic attractor. For $a_1 = 0.55$ (Fig. 4(c)), we plotted two orbits

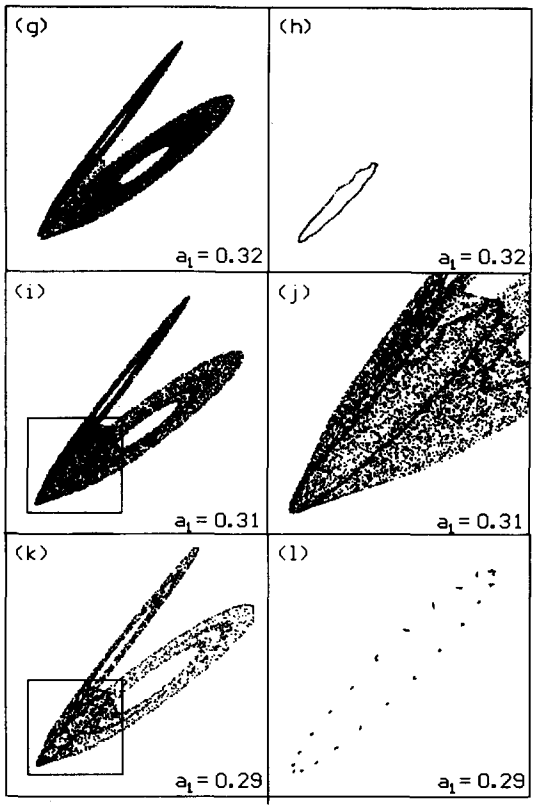


Fig. 4(g)–(l). Horizontal x-axis and vertical y-axis: (g)(h)(i)(k) $80 \leq x, y, \leq 260$, (j)(l) $95 \leq x, y \leq 160$. Parameters: $Y^c = 250$, $I^r = 10$, $U^{aut} = 20$, $m_1 = 0.5$, $m_2 = 0.2$, $m_3 = 0.14$ and $a_2 = 1.31$. Attractors for decreasing values of a_1. (g)–(h) Coexistence of a strange-chaotic attractor in (g) and a stable periodic orbit in (h). (i)–(j) unique strange-chaotic attractor; (j) is an enlargement of the small box in (i). (k)–(l) transient chaotic dynamics in (k) before the orbit settles down to a 23-piece attractor in (l), which is an enlargement of the small box in (k).

with different initial states converging to two different attractors: a stable period 2 orbit (indicated by arrows) and, presumably, a quasi-periodic attractor. Thus, the 3D Hicks model exhibits *coexistence of attractors.* Using Proposition 1 of Subsection 3.1, it follows that the period 2 orbit is created at $a_1 = 0.59$ (cf. (20a)). As a_1 is decreased, for $a_1 \approx 0.39$, the period 2 orbit becomes unstable, with two complex eigenvalues crossing the unit circle. For $a_1 = 0.39$ (Fig. 4(d)), we still have coexistence of two attractors; a quasi-periodic attractor and a second attractor consisting of two parts, as indicated by the arrows in Fig. 4(d). Each part of the second attractor seems to be homeomorphic to a circle with the two points of the projected unstable period 2 orbit lying approximately at the centre of each of the two 'circles'. The largest Lyapunov exponents of both attractors are close to zero; therefore for $a_1 = 0.39$, we have the coexistence of two quasi-periodic attractors. In Figs. 4(c) and 4(d), we have observed the coexistence of two different types of attractors. The first is an attractor with the unstable equilibrium approximately at the centre of the (x, y) projection of

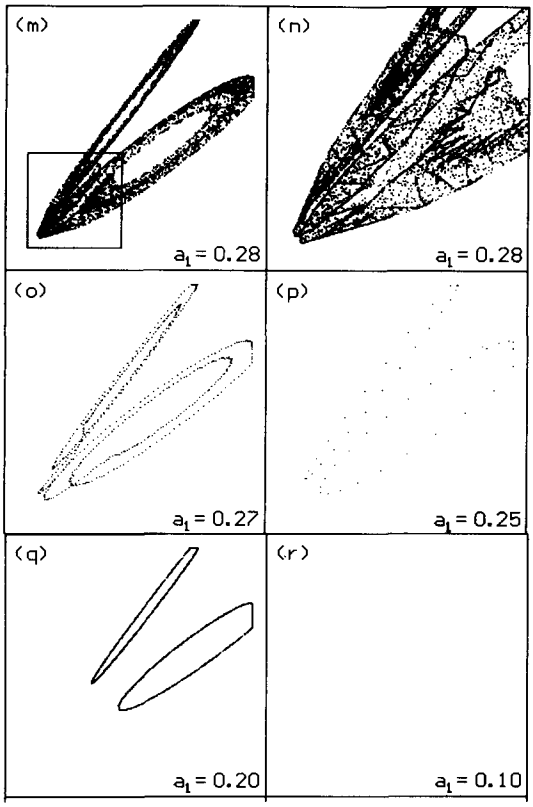


Fig. 4(m)–(r). Horozontal x-axis: $80 \leq x \leq 260$. Vertical y-axis: $80 \leq y \leq 260$. Parameters: $Y^c = 250$, $I^f = 10$, $I^{aut} = 20$, $m_1 = 0.5$, $m_2 = 0.2$, $m_3 = 0.14$ and $a_2 = 1.31$. Attractors for decreasing values of a_1. (m)–(n) unique strange-chaotic attractor; (n) is an enlargement of the small box in (m). (o)–(p) stable periodic orbits. (q) two-piece quasi-attractor. (r) stable period 2 orbit.

the attractor. We call this a *P1-type attractor*. The second is an attractor that consists of two (disjoint) parts. Approximately at the centre of each of the two parts of the projected attractor lies one point of the projected unstable period 2 orbit. We call this a *P2-type attractor*.

The next three diagrams (Figs. 4(e–g)) show how the P2-type attractor changes as a_1 is further decreased. For $a_1 = 0.37$ (Fig. 4(e)), a stable periodic orbit occurs. For $a_1 = 0.35$ (Fig. 4(f)), the largest Lyapunov exponent $\lambda_1 \approx 0.010$, the Lyapunov dimensions $D_L \approx 2.01$ and the correlation dimension $D_C \approx 1.85$. These numerical results indicate that the P2-type attractor has evolved into a strange, chaotic attractor that consists of two parts. For $a_1 = 0.32$ (Fig. 4(g)), we also have a strange, chaotic attractor with $\lambda_1 \approx 0.012$, $D_L \approx 2.02$ and $D_C \approx 2.01$. The diagram suggests that the two parts are connected to one another.[5]

[5] Of course, since the diagrams are (x, y) projections of 3D attractors, from this diagram, one cannot decide whether the 3D attractor really consists of two parts that are not connected to one another, or whether the two parts are connected to one another in the 3D phase space.

We have described how the P2-type attractor has evolved into a strange, chaotic attractor as a_1 is decreased from 0.39 to 0.32. Numerical experiments indicate that, for all a_1 values between 0.39 and 0.32, in addition to this P2-type attractor, there also coexists a P_1-type attractor. For $a_1 = 0.32$, this coexisting attractor is shown in Fig. 4(h). However, for $a_1 = 0.31$, the P1-type attractor suddenly disappears. Numerical simulations indicate that, for $a_1 = 0.31$ (Fig. 4(i) and the enlargement in Fig. 4(j)), a unique strange attractor occurs with the largest Lyapunov exponent $\lambda_1 \approx 0.021$, Lyapunov dimension $D_L \approx 2.03$ and correlation dimension $D_C \approx 1.92$. Our numerical simulations indicate that almost all orbits (in the sense of Lebesque measure) are attracted to this strange attractor. For $a_1 = 0.29$ (Figs. 4(k) and 4(l)), in some sense, the situation is reversed: the P2-type attractor has disappeared and the model has a unique attractor of the P1-type. Fig. 4(k) shows the initial part of an orbit, whereas Fig. 4(l) (on a different scale) shows the same orbit after a long transient time. The diagrams suggest transient chaotic behaviour before the orbit settles down to the P1-type attractor.

For $a_1 = 0.28$ (Fig. 4(m) and the enlargement in Fig. 4(n)), the P1-type attractor has disappeared again, and it seems that we have a unique strange, chaotic attractor of the P2-type with largest Lyapunov exponent $\lambda_1 \approx 0.016$ and correlation dimension $D_C \approx 1.01$.[6] Numerical experiments suggest that, for $0 \leq a_1 \leq 0.28$, the model has a unique attractor of the P2-type, and that the P1-type attractor does not occur anymore. Figs. 4(o–r) show how the P2-type attractor changes as a_1 is further decreased. For $a_1 = 0.27$ (Fig. 4(o)), a stable periodic orbit occurs. The points of this periodic orbit lie on a piecewise linear closed curve with a structure similar to the P2-type attractor. For $a_1 = 0.25$ (Fig. 4(p)), we have a stable period 50 orbit and, for $a_1 = 0.2$ (Fig. 4(q)), a P2-type quasi-periodic attractor. The unstable period 2 orbit lies approximately at the centre of each of the two 'circles' in Fig. 4(q). When a_1 is further decreased, this unstable period 2 orbit, with one point in the linear region D^l and the other point in the floor-region D^f, 'collides' with the 'income ceiling' (the lines $x = Y^c$ and $y = Y^c$), for $a_1 \approx 0.137$. The parameter value a_1 for which this 'collision' occurs can be computed from the parameter condition (20b) in Subsection 3.1. For $a_1 < 0.137$, we have a period 2 orbit, with one point in the floor region and the other in the ceiling region (see Fig. 4(r) for $a_1 = 0.1$). According to stability condition (21a), this period 2 orbit (if it exists) is always stable.

3.3.2. Time series

To obtain more insight into the dynamics, in Fig. 5 we present several income time paths corresponding to the strange-chaotic attractors, in Fig. 4. Figs. 5(a–c) show income time paths for $a_1 = 0.32$, where we have the coexistence of a period attractor (Fig. 4(h)) and a strange-chaotic attractor (Fig. 4(g)). The first is a P1-type attractor and the second a P2-type attractor. The first time path (Fig. 5(a)) converges

[6] Numerical computation of the Lyapunov dimension D_L leads to overflow problems. This might be due to the fact that, for $a_1 = 0.28$, the income ceiling becomes effective and $\lambda_2 = -\infty$, since, in the ceiling region, the map H has a triple eigenvalue 0. If indeed $\lambda_1 > 0$ and $\lambda_2 = -\infty$, then $D_L = 1$.

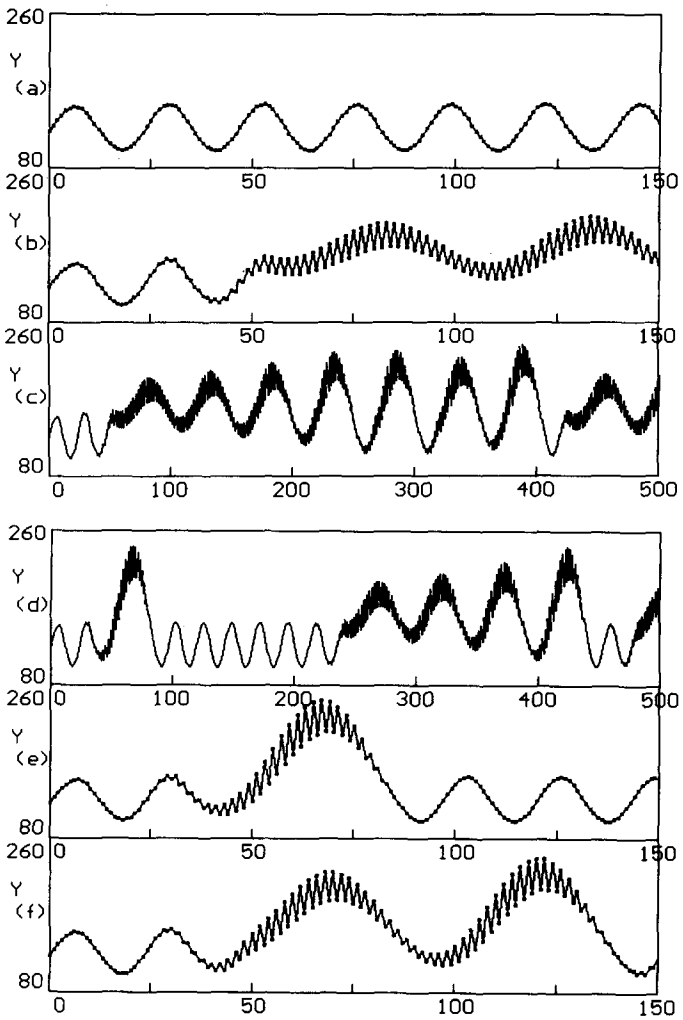


Fig. 5. Time paths of the income Y_t, corresponding to the attractors in Figs. 4(g)–(h) and 4(i)–(j). Vertical axis: $80 \leq Y \leq 260$. Parameters: $Y^c = 250$, $I^f = 10$, $I^{aut} = 20$, $m_1 = 0.5$, $m_2 = 0.2$, $m_3 = 0.14$ and $a_2 = 1.31$; initial state (122.5, 116, 111) in (a)(f) and slightly perturbed initial state in (b)(c)(d)(e). (a)–(c) $a_1 = 0.32$: Coexistence of regular (periodic) and irregular (chaotic) time paths. (d)–(f) $a_1 = 0.31$: single time path exhibiting both regular and irregular behaviour.

to the P1-type attractor, and exhibits regular oscillatory behaviour. In fact, the time path is approximately periodic with a period 23. The second time path (Fig. 5(b)) converges to the P2-type strange, chaotic attractor and is characterized by permanent short, up-and-down oscillations. In Fig. 5(c), a longer part (500 time periods) of the same time path is shown. It is characterized by a slow oscillatory movement under permanent short, up-and-down oscillations. The overall picture of the slow oscillations is irregular. One might say that a short period 2 oscillation is superimposed on to

a long, irregular oscillation. The initial value of the first time path (Fig. 5(a)) is $(x_0, y_0, z_0) = (122.5, 116, 111)$, while the initial value of the second time path (Figs. 5(b) and 5(c)) is obtained by slightly disturbing the initial value (x_0, y_0, z_0), by decreasing the first coordinate x_0 by 1%. Apparently, it depends sensitively on the initial state whether a time path behaves regularly as in Fig. 5(a) or irregularly as in Fig. 5(b). For an arbitrary initial state, it may be difficult to predict whether regular or irregular behaviour will occur.

Figs. 5(d)–(f) show income time paths for $a_1 = 0.31$. In that case, a unique strange, chaotic attractor occurs as shown in Fig. 4(i). Fig. 5(d) shows the first 500 time periods of the income time path with the same perturbed initial state as in Fig. 5(c). The time path is characterized by a mixture of the regular behaviour corresponding to the P1-type attractor, and the irregular behaviour corresponding to the P2-type attractor. Phases of approximate period 23 oscillations are interrupted by phases of slow oscillatory behaviour under permanent short up and down oscillations. In fact, the single time path in Fig. 5(d) exhibits both types of behaviour of the two different time paths in Figs. 5(a) and 5(b). Observe that the length of the regular phases varies a lot: in the beginning there are two period 23 oscillations; later on, six consecutive period 23 oscillations; and at the end, one period 23 oscillation. The first 150 periods of the same time path are shown in Fig. 5(e), while Fig. 5(f) shows 150 periods of the time path with the initial state (122.5, 116, 111). Although the initial states are close, the two time paths are quite different, indicating sensitive dependence on initial conditions. We conclude that the chaotic time paths are characterized by a mixture of regular and irregular behaviour. The length of the regular and irregular phases and the moment at which the time path switches from regular to irregular behaviour are difficult to predict.

In the time series in Fig. 5(d), apparently 'regime shifts' occur where the behaviour of income suddenly changes. Recently, Hamilton (1989) explained such regime shifts by exogenous shocks. Alternatively, our results show that simple non-linear models can provide an endogenous explanation of regime shifts without any external shocks. In the 3D Hicks model, for $a_1 = 0.31$ the regime shifts are caused by a strange attractor generating oscillatory behaviour interrupted by bursts of erratic behaviour.

3.3.3. Historical note

It is remarkable that Hicks (1950) has already touched upon the possibility of permanent up and down oscillations. In particular, Hicks presented some time paths exhibiting this type of behaviour (Figs. 9 and 10 in Hicks, 1950, pp. 76–79). Furthermore, Allen (1965, pp. 232–237) considered the possibility of permanent up and down oscillations in the linear multiplier–accelerator model with postponed or distributed induced investment. Both Hicks and Allen noted that the possibility of up and down oscillations may arise when the first partial investment coefficient a_1 is small compared with the second partial investment coefficient a_2. These up-and-down oscillations are closely related to the existence of a period 2 orbit in the Hicks model. Indeed, this period 2 orbit only exists when a_1 is smaller than a_2, cf. parameter condition (20a). The permanent up-and-down oscillations are explained by the occurrence of a P2-type attractor, which may be a stable period 2 orbit, a periodic

or quasi-periodic attractor, or a strange chaotic attractor. Our results show that a P2-type attractor may occur when induced investment is distributed over the two time periods following the change in income, with most investment in the latter period. Hicks did not pay much attention to the phenomenon of permanent up-and-down oscillations. However, this phenomenon seems to be closely related to the occurrence of complicated dynamics in Hicks' non-linear trade cycle model.

4. Concluding remarks

We have reconsidered the dynamics of Hicks' non-linear trade cycle model. Even in the simplest version of the model (the basic Hicks model), a time path does not always converge to a periodic time path. In addition to stable periodic behaviour, the basic Hicks model also exhibits quasi-periodic dynamics. However, the dynamics of the basic Hicks model are always regular. The income fluctuations are either periodic or almost periodic (quasi-periodic). Irregular, chaotic fluctuations do not occur in the simplest version of Hick's non-linear trade cycle model.

In the basic Hicks model, all consumption lags one period behind income, while all induced investment is concentrated in the period immediately following the originating change in income. When consumption or investment is distributed over different time periods, the regularity in the dynamics of the model will not persist. Extra time lags in the consumption or the investment equation may cause complicated, chaotic dynamics. This simple example shows that the introduction of extra time lags, which is often more realistic, may lead to considerable complications of the dynamic behaviour of a model.

When consumption is distributed over three periods following income, and induced investment is distributed over two periods following the change in income, the Hicks model is a 3D piecewise linear model. The 3D Hicks model is the simplest version of the model, after the basic Hicks model. Our numerical investigations indicate that the 3D Hicks model can have strange, chaotic attractors. Even when the largest fraction of income is consumed in the first period following income, chaotic dynamics cannot be excluded. In particular, with most consumption in the first period and less consumption in the third period, a relatively large proportion of induced investment in the second period following the change in income may cause chaotic dynamics. Hence, in the case that seems to be the most relevant from an economic point of view, chaotic dynamics is a possibility not to be dismissed lightly.

The strange attractors in the Hicks model sometimes exhibit almost periodic behaviour interrupted by sudden bursts of erratic behaviour. Thus, the Hicks model provides an endogenous explanation of regime shifts without any exogenous shocks. Another source of complex dynamics in the Hicks model is the coexistence of different attractors. For example, in the presence of a small amount of noise, the coexistence of two stable periodic orbits can generate very erratic dynamical behaviour.

We believe that, from a numerical point of view, we understand the dynamics of the 3D Hicks model reasonably well. However, from a theoretical point of view, several open questions remain. In particular, the sudden global changes in the

dynamics of the model, i.e. the sudden appearance or disappearance of a certain attractor of the model, is not completely understood. Apparently, so-called homoclinic and heteroclinic bifurcations cause these global changes. In particular, the phenomenon that two different, coexisting attractors 'melt' into one single attractor when a parameter is varied might be explained by a heteroclinic bifurcation. This remains to be investigated.

The Hicks model is a prototype of a model with a very simple type of non-linearity: natural 'ceilings' (upper bounds) and 'floors' (lower bounds) imposed on an (unstable) linear system. Within these upper and lower bounds, all relationships between the variables are linear; outside the bounds, the corresponding variable is constant, attaining its maximum or minimum value. In particular, for economics, this type of non-linearity may be important. As we have seen, even this very simple type of non-linearity may lead to strange dynamics. Trying to fit models with 'Hicksian' non-linearities to economic data seems to be a promising topic for future research.[7] In this way, one might try to explain a significant part of observed economic fluctuations.

Acknowledgements

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Appendix A

This appendix contains the main results concerning the dynamics of the basic two-dimensional Hicks model; proofs can be found in Hommes (1991). According to the first result, the piecewise linear map H (8), generating the dynamics of the basic Hicks model, has a one-dimensional attracting set.

Theorem 1. *If* $a > 1$, *then the equilibrium* E^1 *is unstable, and the map* H *in* (8) *has an attracting set* A, *which is a piecewise linear closed curve. All orbits, except the unstable equilibrium* E^1, *are attracted to the set* A.

[7] During the revision of this paper, I became aware that this approach has been followed recently by Pesaran and Potter (1994).

Concerning the dynamics on the attracting set A, we have the following results:

Theorem 2. *Let H be the map in* (8), *and A the attracting set of H that is homeomorphic to a circle. For the dynamics on the attracting set A, we have one of the following:*

(a) *The map* H: $A \to A$ *is topologically conjugate to a piecewise linear* (*orientation preserving*) *circle homeomorphism.*

(b) *The map* H: $A \to A$ *is topologically conjugate to a piecewise linear, non-decreasing circle map f of degree 1. There exists a unique circle arc I such that f is constant on I, while f is strictly increasing on the complement of I.*

An important corollary of Theorem 2 is as follows.

Corollary 1. *Each time path in the elementary Hicks model, except for the equilibrium* E^1, *rotates with a unique average rotation* (*the rotation number*) *around the unstable equilibrium.*

When the rotation number is rational, then the Hicks model has a periodic attractor. However, the rotation number can be irrational. In particular; as follows.

Corollary 2. *There exist parameters for which the basic Hicks model has a quasi-periodic attractor, attracting all orbits except the equilibrium.*

In summary, the basic Hicks model exhibits both periodic and quasi-periodic dynamics. Chaotic attractors do not occur in the simplest version of Hicks' non-linear trade cycle model.

Appendix B

In this appendix, we prove Proposition 1 of Subsection 3.1, concerning the existence of a period 2 orbit in the 3D Hicks model.

Proof of proposition 1.

First, we assume that the map H has a period 2 orbit $\{(x_0, y_0, z_0),(x_1, y_1, z_1)\}$, with (x_0, y_0, z_0) in the linear region D^l and (x_1, y_1, z_1) in the floor region D^f. Using the expressions for the floor map H^f in (17a) and the multiplier–accelerator map H^l in (17b), it follows that x_0, y_0, z_0, x_1, y_1 and z_1 satisfy the following set of linear equations:

$$x_1 = m_1 x_0 + m_2 y_0 + m_3 z_0 + a_1(x_0 - y_0) + a_2(y_0 - z_0) + I^{aut}, \tag{26a}$$

$$y_1 = x_0, \tag{26b}$$

$$z_1 = y_0, \tag{26c}$$

$$x_0 = m_1 x_1 + m_2 y_1 + m_3 z_1 + I^{aut} - I^f, \tag{26d}$$

$$y_0 = x_1, \tag{26e}$$

$$z_0 = y_1. \tag{26f}$$

The variables x_1, y_1, z_1 and z_0 can be expressed in x_0 and y_0 as follows: $x_1 = y_0$, $y_1 = x_0$, $z_1 = y_0$ and $z_0 = y_1 = x_0$. By substituting these values of x_1, y_1, z_1 and z_0 into (26a) and (26d), this set of six linear equations is reduced to the following set of two linear equations for x_0 and y_0.

$$y_0 = m_1 x_0 + m_2 y_0 + m_3 x_0 + a_1(x_0 - y_0) + a_2(y_0 - x_0) + I^{\text{aut}}, \tag{27a}$$

$$x_0 = m_1 y_0 + m_2 x_0 + m_3 y_0 + I^{\text{aut}} - I^{\text{f}}. \tag{27b}$$

Solving these linear equations for x_0 and y_0 yields:

$$x_0 = \frac{I^{\text{aut}} - I^{\text{f}}}{1 - m_1 - m_2 - m_3} + \frac{I^{\text{f}}(m_1 + m_3)}{(1 - m_1 - m_2 - m_3)(1 + m_1 - m_2 + m_3 + a_1 - a_2)}, \tag{28a}$$

$$y_0 = \frac{I^{\text{aut}} - I^{\text{f}}}{1 - m_1 - m_2 - m_3} + \frac{I^{\text{f}}(1 - m_2)}{(1 - m_1 - m_2 - m_3)(1 + m_1 - m_2 + m_3 + a_1 - a_2)}. \tag{28b}$$

Hence, if H has a period 2 orbit with one point in D^1 and one point in D^{f}, then the period 2 orbit is given by $\{(x_0, y_0, x_0),(y_0, x_0, y_0)\}$, where x_0 and y_0 are given by (28a) and (28b), with (x_0, y_0, x_0) in D^1 and (y_0, x_0, y_0) in D^{f}.

Recall that a point (x, y, z) lies in D^{f} if and only if $a_1(x - y) + a_2(y - z) < -I^{\text{f}}$ and $Y^{\text{f}} \leq x, y, z \leq Y^{\text{c}}$, where the floor level of income was defined as $Y^{\text{f}} = (I^{\text{aut}} - I^{\text{f}})/(1 - m_1 - m_2 - m_3)$. Since $1 - m_2 > 0$ and $1 - m_1 - m_2 - m_3 > 0$, the conditions $x_0, y_0 \geq Y^{\text{f}}$, with x_0 and y_0 as given in (28a) and (28b), yield $1 + m_1 - m_2 + m_3 + a_1 - a_2 > 0$, or, equivalently, $a_2 - a_1 < 1 + m_1 + m_2 + m_3$. Furthermore, the fact that (y_0, x_0, y_0) lies in D^{f} implies the following two inequalities:

(C1) $\quad a_1(y_0 - x_0) + a_2(x_0 - y_0) < -I^{\text{f}}$,

(C2) $\quad x_0, y_0 \leq Y^{\text{c}}$.

Substituting the expressions for x_0 and y_0 of (28a) and (28b) into (C1) and dividing by I^{f} yields $(a_1 - a_2)/(1 + m_1 - m_2 + m_3 + a_1 - a_2) < -1$. Since $(1 + m_1 - m_2 + m_3 + a_1 - a_2) > 0$, this inequality is equivalent to $a_2 - a_1 > (1 + m_1 - m_2 + m_3)/2$, proving condition (20a).

Observe that $y_0 > x_0$. Simple computations show that $y_0 < Y^{\text{c}}$ is equivalent to

$$I^{\text{aut}} + I^{\text{f}}\left[\frac{(a_2 - a_1 - m_1 - m_3)}{(1 + m_1 - m_2 + m_3 + a_1 - a_2)}\right] < (1 - m_1 - m_2 - m_3)Y^{\text{c}},$$

proving (20b).

Now assume that conditions (20a) and (20b) are satisfied. Define x_0 and y_0 by (28a) and (28b). The computations above show that conditions (20a) and (20b) imply

that the point (y_0, x_0, y_0) lies in D^f. If (y_0, x_0, y_0) lies in D^f, then

$$a_1(y_0 - x_0) + a_2(x_0 - y_0) < -I^f$$

or equivalently,

$$a_1(x_0 - y_0) + a_2(y_0 - x_0) > I^f,$$

implying that (x_0, y_0, x_0) does not lie in D^f. Recall that a point (x, y, z) lies in D^c if $H^1(x, y, z) > Y^c$. Since $H^1(x_0, y_0, x_0) = (y_0, x_0, y_0)$ and $y_0 < Y^c$ from condition (20b), it follows that (x_0, y_0, x_0) does not lie in D^c either, so the points (x_0, y_0, x_0) lies in D^1. Furthermore, the points $(x_0, y_0, z_0) = (x_0, y_0, x_0)$ and $(x_1, y_1, z_1) = (y_0, x_0, y_0)$ satisfy the set of linear equations (26a)–(26f). We conclude that $\{(x_0, y_0, z_0),(x_1, y_1, z_1)\}$ is a period 2 orbit of the map H. □

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